

New extensions to the sumsets with polynomial restrictions

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Abstract

By taking the leading and the second leading coefficients of the Morris identity, we get new polynomial coefficients. These coefficients lead to new results in the sumsets with polynomial restrictions by the polynomial method of N. Alon.

1 Introduction

Let A_1, \dots, A_n be finite subsets of a field F with $0 < k_1 = |A_1| \leq \dots \leq k_n = |A_n|$, where the characteristic of F is infinite or a prime. We are concerning the lower bounds for the following sumsets:

$$\{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, \text{ } a_i' \text{'s satisfy certain restrictions}\}.$$

In 1999, N. Alon got the following lemma which is called the Combinatorial Nullstellensatz.

Lemma 1.1. (Alon [1]) *Let A_1, \dots, A_n be finite subsets of a field F with $|A_i| > k_i$ for $i = 1, \dots, n$, where k_1, \dots, k_n are nonnegative integers. If the coefficient of the monomial $x_1^{k_1} \dots x_n^{k_n}$ in $f(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$ is nonzero and $k_1 + \dots + k_n$ is the total degree of f , then there are $a_1 \in A_1, \dots, a_n \in A_n$ such that $f(a_1, \dots, a_n) \neq 0$.*

The Combinatorial Nullstellensatz soon led to many results [5, 6, 8, 10–14] in the sumsets. It also implies a polynomial method in finding the lower bounds for various restricted sumsets, which is the following lemma.

Lemma 1.2. (Alon et al. [2, 3]). *Let A_1, \dots, A_n be finite nonempty subsets of a field F with $|A_i| = k_i$ for $i = 1, \dots, n$. Let $P(x_1, \dots, x_n) \in F[x_1, \dots, x_n] \setminus \{0\}$ and $\deg P \leq \sum_{i=1}^n (k_i - 1)$. If the coefficient of the monomial $x_1^{k_1-1} \dots x_n^{k_n-1}$ in the polynomial*

$$P(x_1, \dots, x_n)(x_1 + \dots + x_n)^{\sum_{i=1}^n (k_i - 1) - \deg P}$$

does not vanish, then we have

$$|\{a_1 + \dots + a_n : a_i \in A_i, P(a_1, \dots, a_n) \neq 0\}| \geq \sum_{i=1}^n (k_i - 1) - \deg P + 1.$$

Throughout this paper, we need the following definitions and notations.

Let k, m be nonnegative integers and let n be a positive integer. Denote T by a nonempty subset of $\{1, \dots, n\}$ and denote F by a field with characteristic p (infinite or a prime). For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} be subsets of F . Define

- Condition (a): $|A_i| = k, |S_{ij}| \leq 2m$.
- Condition (b): $|A_i| = k - n + i, |S_{ij}| < 2m$.

By using an equivalent form of Lemma 1.2 (Proposition 2.1 of [5]), Q. H. Hou and Z. W. Sun got the following theorem.

Theorem 1.3. (Hou and Sun [5]) *For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (a). If $p > \max\{mn, (k-1)n - mn(n-1)\}$, then for the set*

$$C = \{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j\}, \quad (1.1)$$

we have

$$|C| \geq (k + m - mn - 1)n + 1.$$

In the original paper of Q. H. Hou and Z. W. Sun [5], they restrict $|S_{ij}| \leq m$, but the result is still right when $|S_{ij}|$ are extended to $2m$, as pointed out by Z. W. Sun in [11].

After that, Z. W. Sun and Y. N. Yeh got a theorem very closely to Theorem 1.3.

Theorem 1.4. (Sun and Yeh [13]) *For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (b). If $p > \max\{mn, (k-1)n - mn(n-1)\}$, then we have*

$$|C| \geq (k + m - mn - 1)n + 1,$$

where C is defined as in (1.1).

In this paper, we will show that Theorem 1.3 and Theorem 1.4 are actually equivalent by a relation between two kinds of polynomial coefficients. By getting new polynomial coefficients we obtain some further results related to Theorem 1.3 and Theorem 1.4.

By using the leading coefficient of the Morris identity [7], we get the following theorem.

Theorem 1.5. *For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (a), and let $\sum_{i \in T} \alpha_i \neq 0$ for $\alpha_i \in F$. If $p > \max\{mn, (k-1)n - mn(n-1) - 1\}$, then for the set*

$$C = \{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j \text{ and } \sum_{i \in T} \alpha_i a_i \neq 0\}, \quad (1.2)$$

we have

$$|C| \geq (k + m - mn - 1)n.$$

By a relation between two kinds of polynomial coefficients in Section 2, we get the following theorem corresponds to Theorem 1.5.

Theorem 1.5'. For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (b), and let C, α_i and p be as in Theorem 1.5. We have

$$|C| \geq (k + m - mn - 1)n. \quad (1.3)$$

By using the second leading coefficient of the Morris identity, we can get the following theorems.

Theorem 1.6. For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (a), and let $\sum_{i \in T} \alpha_i \neq 0$ for $\alpha_i \in F$. If $p > \max\{mn, (k-1)n - mn(n-1) - 2\}$, then for the set

$$C = \{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i - a_j \notin S_{ij} \text{ if } i \neq j \text{ and } \sum_{i \in T} \alpha_i a_i^2 \neq 0\}, \quad (1.4)$$

we have

$$|C| \geq (k + m - mn - 1)n - 1.$$

Theorem 1.6'. For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (b), and let C, α_i and p be as in Theorem 1.6. We have

$$|C| \geq (k + m - mn - 1)n - 1.$$

Theorem 1.7. For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (a), and let $\sum_{i,j \in T} \alpha_{ij} \neq 0$ for $\alpha_{ij} \in F$. If $p > \max\{mn, (k-1)n - mn(n-1) - 2\}$, then for the set

$$C = \{a_1 + \dots + a_n : a_1 \in A_1, \dots, a_n \in A_n, a_i - a_j \notin S_{ij} \text{ and } \sum_{i \in T} \alpha_{ij} a_i a_j \neq 0, \text{ if } i \neq j\}, \quad (1.5)$$

we have

$$|C| \geq (k + m - mn - 1)n - 1. \quad (1.6)$$

Theorem 1.7'. For $i, j = 1, \dots, n$ with $i \neq j$, let A_i and S_{ij} satisfy the Condition (b), and let C, α_{ij} and p be as in Theorem 1.7. We have

$$|C| \geq (k + m - mn - 1)n - 1. \quad (1.7)$$

This paper is organized as follows. In Section 2 we give a relation between two kinds of polynomial coefficients. In Section 3 we obtain new polynomial coefficients, which are the keys to the polynomial method by N. Alon. In Section 4 we prove Theorem 1.5 with the polynomial method, and the proof of other theorems are routine according to the proof of Theorem 1.5.

2 A relation between two kinds of polynomial coefficients

Let

$$H'_m(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} \cdot L(x_1, \dots, x_n), \quad (2.1)$$

and

$$H_m(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} \cdot L(x_1, \dots, x_n), \quad (2.2)$$

where $L(x_1, \dots, x_n)$ is a symmetric polynomial in the x 's.

We have the following relation between the coefficients of $H'_m(x)$ and $H_m(x)$.

Lemma 2.1.

$$[x_1^{k-1} \dots x_n^{k-1}] H'_m(x) = n! [x_1^{k-n} \dots x_n^{k-1}] H_m(x), \quad (2.3)$$

where $[x_1^{j_1} \dots x_n^{j_n}] P(x_1, \dots, x_n)$ denotes the coefficient of the monomial $x_1^{j_1} \dots x_n^{j_n}$ in the polynomial $P(x_1, \dots, x_n)$.

Proof. By the formula of $H'_m(x)$ in (2.1), we have

$$[x_1^{k-1} \dots x_n^{k-1}] H'_m(x) = [x_1^{k-1} \dots x_n^{k-1}] \prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} L(x_1, \dots, x_n). \quad (2.4)$$

It is well-known that

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{w \in S_n} (\text{sgn } w) \prod_{l=1}^n x_{w(l)}^{n-l},$$

where S_n is the symmetric group of all permutations on $\{1, \dots, n\}$ and $\text{sgn } w$ equals 1 or -1 according to whether w is even or odd. Since $\prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} L(x_1, \dots, x_n)$ is antisymmetric in the x 's, we have

$$\begin{aligned} [x_1^{k-1} \dots x_n^{k-1}] H'_m(x) &= [x_1^{k-1} \dots x_n^{k-1}] \sum_{w \in S_n} (\text{sgn } w) \prod_{l=1}^n x_{w(l)}^{n-l} \\ &\quad \cdot (\text{sgn } w) \prod_{1 \leq i < j \leq n} (x_{w(i)} - x_{w(j)})^{2m-1} L(x_{w(1)}, \dots, x_{w(n)}) \\ &= [x_1^{k-1} \dots x_n^{k-1}] \sum_{w \in S_n} \prod_{l=1}^n x_{w(l)}^{n-l} \prod_{1 \leq i < j \leq n} (x_{w(i)} - x_{w(j)})^{2m-1} L(x_{w(1)}, \dots, x_{w(n)}). \end{aligned} \quad (2.5)$$

Since the monomial $x_1^{k-1} \dots x_n^{k-1}$ is symmetric in the x 's, (2.5) becomes

$$\begin{aligned} [x_1^{k-1} \dots x_n^{k-1}] H'_m(x) &= n! [x_1^{k-1} \dots x_n^{k-1}] \prod_{l=1}^n x_l^{n-l} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} L(x_1, \dots, x_n) \\ &= n! [x_1^{k-n} \dots x_n^{k-1}] H_m(x). \end{aligned}$$

□

Note that this lemma can be extended to the q -cases, see J. Stembridge [9, Theorem 4.1].

3 Auxiliary propositions

Denote

$$\Delta = \prod_{l=0}^{n-1} \frac{(m(l+1))!}{(b+ml)!m!}. \quad (3.1)$$

By taking the leading and the second leading coefficients of the Morris identity [7], we obtained the following identities.

Proposition 3.1. (Gessel et al. [4]). *For $n, b, m \in \mathbb{N}$, we have*

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] \left(\sum_{i=1}^n x_i \right)^{nb} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} = (-1)^{m \binom{n}{2}} (nb)! \Delta, \quad (3.2)$$

and for $1 \leq r \leq n$,

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] x_r^2 \left(\sum_{i=1}^n x_i \right)^{nb-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} = (-1)^{m \binom{n}{2}} b(b-m(n-1)-1)(nb-2)! \Delta. \quad (3.3)$$

If we let $b+m(n-1) = k-1$, then (3.2) becomes the key identity in [5, Proposition 2.2] of Q. H. Hou and Z. W. Sun. The identity led to Theorem 1.3 by the polynomial method (Lemma 1.2). On the other hand, by applying Lemma 2.1 to (3.2) we can get Corollary 2.2 in [13] of Z. W. Sun and Y. N. Yeh, which led to Theorem 1.4 by Lemma 1.2. Thus Theorem 1.3 and Theorem 1.4 are equivalent in the view of Lemma 2.1.

From (3.2) and (3.3) we can deduce the following identities.

Proposition 3.2. *For $1 \leq r \leq n$, we have*

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] x_r \left(\sum_{i=1}^n x_i \right)^{nb-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} = (-1)^{m \binom{n}{2}} b(nb-1)! \Delta. \quad (3.4)$$

For $1 \leq i \neq j \leq n$, we have

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] x_i x_j \left(\sum_{i=1}^n x_i \right)^{nb-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} = (-1)^{m \binom{n}{2}} b(b+m)(nb-2)! \Delta. \quad (3.5)$$

Proof. Rewrite (3.2) as

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n x_i \right)^{nb-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} = (-1)^{m \binom{n}{2}} (nb)! \Delta.$$

Therefore by the symmetry of the x 's we can get (3.4).

Rewrite the left hand side of (3.2) as

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] \left(\sum_{i=1}^n x_i \right)^2 \left(\sum_{i=1}^n x_i \right)^{nb-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m}.$$

Expand $(\sum_{i=1}^n x_i)^2$ and by the symmetry of the x 's, for $1 \leq i \neq j \leq n$ and $1 \leq r \leq n$ the above equation becomes

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)} \right] (nx_r^2 + n(n-1)x_i x_j) \left(\sum_{i=1}^n x_i \right)^{nb-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m}.$$

It leads to (3.5) by substituting (3.3) into the above equation. \square

By (3.3)–(3.5) and Lemma 2.1 we can get the following identities.

Proposition 3.3. *For $1 \leq r \leq n$ and $1 \leq i \neq j \leq n$, we have*

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)-n+l} \right] x_r \left(\sum_{i=1}^n x_i \right)^{nb-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} = (-1)^{m \binom{n}{2}} \frac{b(nb-1)!}{n!} \Delta, \quad (3.6)$$

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)-n+l} \right] x_r^2 \left(\sum_{i=1}^n x_i \right)^{nb-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} = (-1)^{m \binom{n}{2}} \frac{b(b-m(n-1)-1)(nb-2)!}{n!} \Delta, \quad (3.7)$$

$$\left[\prod_{l=1}^n x_l^{b+m(n-1)-n+l} \right] x_i x_j \left(\sum_{i=1}^n x_i \right)^{nb-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m-1} = (-1)^{m \binom{n}{2}} \frac{b(b+m)(nb-2)!}{n!} \Delta. \quad (3.8)$$

4 Proof of the theorems

Our proof of Theorem 1.5 follows the same line as Z. W. Sun and Y. N. Yeh's proof of Theorem 1.4.

Proof of Theorem 1.5. The case $n = 1$ or $k - 1 < m(n - 1)$ is trivial. So we assume $n \geq 2$ and $b = k - 1 - m(n - 1) \geq 0$.

Since $|F| \geq p > mn \geq 2m$, we can extend each $S_{ij} (1 \leq i < j \leq n)$ to a subset S_{ij}^* of F with cardinality $2m$. By Lemma 1.2 it suffices to show that

$$[x_1^{k-1} \cdots x_n^{k-1}] \left(\sum_{i \in T} \alpha_i x_i \right) \left(\sum_{i=1}^n x_i \right)^{nb-1} \prod_{1 \leq i < j \leq n} \prod_{c \in S_{ij}^*} (x_i - x_j + c)$$

does not vanish. Let e denote the multiplicative identity of the field F . Then the above coefficient equals he where

$$h = [x_1^{k-1} \cdots x_n^{k-1}] \left(\sum_{i \in T} \alpha_i x_i \right) \left(\sum_{i=1}^n x_i \right)^{nb-1} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2m} \in \mathbb{Z}.$$

By (3.4),

$$h = \left(\sum_{i \in T} \alpha_i \right) (-1)^{m \binom{n}{2}} b(nb-1)! \prod_{l=0}^{n-1} \frac{(m(l+1))!}{(b+ml)!m!}.$$

As $p > mn, p > nb - 1$ and $\sum_{i \in T} \alpha_i \neq 0$, p does not divide h and hence $he \neq 0$. This concludes the proof. \square

We notice that recently Z. W. Sun and L. L. Zhao got a theorem [14, Theorem 1.3] to count the lower bounds for the sumsets with general polynomial restrictions, but it seems that their theorem is not appropriate in getting Theorem 1.5 directly.

The proofs of other theorems in this paper are routine by applying Lemma 1.2 to (3.3), (3.5) and Proposition 3.3.

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